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RESEARCH ON NONLINEAR DIFFERENTIAL EQUATIONS

FINAL REPORT

ROBERT H. MARTIN, JR.

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20. ABSTRACT

The research described in this report deals with the existence and behavior of solutions to nonlinear differential equations in a Banach space. This includes the determination of invariant sets for solutions, relations involving inequalities between solutions, and global existence and asymptotic properties (stability) of solutions. The abstract techniques and results developed of differential equations in Banach spaces are applied to semilinear parabolic systems. These examples indicate that several classical as well as new results in the study of solutions to semilinear partial differential equations may be obtained by abstract techniques.

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RESEARCH ON NONLINEAR DIFFERENTIAL EQUATIONS

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The main topic of mathematical research investigated in this project is the study of the existence and behavior of solutions to semilinear differential equations in Banach spaces. The principle application of these abstract techniques is found in the study of systems of semilinear parabolic equations. The literature generated by this project includes the published article [3]*; the submitted articles [2], [4]; the preliminary report [6]; and the book [5].

The results of [3] characterize invariant sets for evolution systems in Banach spaces. So let X be a real or complex Banach space with norm denoted $|\cdot|$ and let D be a closed subset of X. A family $U = \{U(t,s) : t \geq s \geq 0\}$ of mappings from D into D is called an evolution system of type ω $(\omega \in \mathbb{R})$ if each of the following conditions hold:

(U1) $U(t,t)x = x$ for all $t \geq 0$, $x \in D$.

(U2) $U(t,s)U(s,r)x = U(t,r)x$ for all $t \geq s \geq r \geq 0$, $x \in D$.

(U3) $|U(t,s)x - U(t,s)y| \leq |x - y|e^{\omega(t-s)}$ for all $t \geq s \geq 0$, $x,y \in D$.

(U4) The map $(t,s) \to U(t,s)x$ of $\{(t,s) : t \geq s \geq 0\}$ into X is continuous for each $x \in D$.

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*Numbers in brackets refer to the bibliography.

The principle result of this paper is

Theorem 1 [3]. Suppose that U satisfies (U1)-(U4) and that C is a closed subset of X and $C \subset D$. Then these are equivalent

(i) $U(t,s) : C \to C$ for all $t \geq s \geq 0$ (i.e., $x \in C$ and $s \geq 0$ implies that $U(t,s)x \in C$ for all $t \geq s$).

(ii) $\inf_{h \to 0+} \inf d(U(t + h, t)x; C)/h = 0$ for all $t \geq 0$ and $x \in C$, where $d(y;C) \equiv \inf\{|y - z| : z \in C\}$ for each $y \in X$.

Some applications of this result to the study of the behavior of solutions to systems of semilinear parabolic problems is included in [3, §4]. Recently there has been a (perhaps surprising) connection established between Theorem 1 (and its proof) and a number of diverse results in nonlinear functional analysis (fixed points theorems, normal solvability theorems, and generalizations of the Bishop-Phelps theorem for the existence of tangent functionals for convex sets). A unification of these ideas and techniques can be found in a recent paper of Brezis and Browder [1].

One major topic developed from the study in this project is that of the determination of invariant sets for solutions to semilinear equations. We state the fundamental abstract result first and then indicate its applicability by considering a nonlinear system of parabolic equations. Suppose that X is a Banach space with norm denoted by $|\cdot|$ and that $T = \{T(t,s) : t \geq s \geq 0\}$ is a family of bounded linear operators from X into X that satisfy the following properties:

(T1) $T(t,t) = I$ and $T(t,s)T(s,r) = T(t,r)$ for all $t \geq s \geq r \geq 0$.

(T2) The map $(t,s) \to T(t,s)x$ is continuous from $\{(t,s) : t \geq s \geq 0\}$ into X for each $x \in X$.

Suppose also that D is a closed subset of $[0,\infty) \times X$ and that if $D_t \equiv \{x \in X : (t,x) \in D\}$, then D_t is nonempty for all $t \geq 0$. Now let B be a continuous function from D into X and α a continuous function from $[0,\infty)$ into X. For each $(s,z) \in D$ we consider the existence of solutions to the integral equation

$$\text{(IE)} \qquad u(t) = T(t,s)(z - \alpha(s)) + \alpha(t) + \int_s^t T(t,r)B(r, u(r))dr, \quad t \geq s.$$

If $\delta > 0$, a continuous function $u : [s, s + \delta) \to X$ is said to be a solution to (IE) on $[s, s + \delta)$ if $(t, u(t)) \in D$ for all $t \in [s, s + \delta)$ and u satisfies (IE) for each $t \in [s, s + \delta)$. We have the following basic result on the local existence of solutions to (IE) (recall that $d(x;E) \equiv \inf\{|x - y| : y \in E\}$ for each $x \in X$ and $E \subset X$).

<u>Theorem 2 [4]</u>. In addition to the conditions enumerated in the above paragraph suppose that either

(a) $T(t,s)$ is compact for each $t > s \geq 0$

or

(b) there is a continuous, increasing function $L : [0,\infty) \to [0,\infty)$ such that $|B(t,x) - B(s,y)| \leq L(t)(|t - s| + |x - y|)$ for all $(t,x), (t,y) \in D$ with $t \geq s$.

Then the following statements are equivalent:

(i) For each $(s,z) \in D$ there is a $\delta = \delta(s,z) > 0$ such that (IE) has a solution u on $[s, s + \delta)$.

(ii) $\liminf_{h \to 0+} d(T(t + h, t)(z + \alpha(t)) + \alpha(t + h) + hB(t,z); D_{t+h})/h = 0$ for all $(t,z) \in D$.

Investigations concerning the global existence of solutions to (IE) may be carried out using the usual techniques. In fact, if (b) in Theorem 2 holds and also (ii) is satisfied, then each solution to (IE) is unique and the noncontinuable solution exists on $[s,\infty)$ for each $(s,z) \in D$. Several basic results along the lines of Theorem 2 may be found in the book [5, Chapter VIII]. Also, this book contains several examples illustrating these ideas (see Chapter VIII and IX of [5]).

In order to illustrate the applicability of Theorem 2, we consider a system of two semilinear parabolic equations. So suppose that Ω is a smooth bounded domain in $\mathbb{R}^n$, that f and g are continuous functions from $\mathbb{R}^2$ into $\mathbb{R}$, that Δ is the Laplacian operator on Ω, and consider the system

$$\text{(PS)}\quad \begin{cases} u_t(t,x) = a\Delta u(t,x) + f(u(t,x),\, v(t,x)) & \\ u_t(t,x) = b\Delta v(t,x) + g(u(t,x),\, v(t,x)) & \text{for all } t > 0,\ x \in \Omega \\ u(0,x) = u_o(x), \quad v(0,x) = v_o(x) & y \in \partial\Omega \\ u(t,y) = \beta(y), \quad v(t,y) = \gamma(y) & \end{cases}$$

where $a,b > 0$, $u_o, v_o : \bar{\Omega} \to \mathbb{R}$ are continuous, and $\beta,\gamma : \partial\Omega \to \mathbb{R}$ are continuous. The existence and behavior of solutions to the parabolic system can be effectively studied by using abstract techniques, and we give a brief indication of these ideas here (see also [4] and [5, Chapter VIII and IX]). Suppose that $1 < p < \infty$ and that X is the Banach space $L^p \equiv L^p(\Omega;\, \mathbb{R}^2)$ of all measurable functions $\phi = (\phi_1, \phi_2) : \Omega \to \mathbb{R}^2$ such that

$$\|\phi\|_p \equiv [\int_\Omega |\phi(x)|^p dx]^{1/p} < \infty$$

(here $|\cdot|$ is some norm on $\mathbb{R}^2$). In this case, the "linear part" of equation (PS) is autonomous, so the linear evolution system $T = \{T(t,s) : t \geq s \geq 0\}$ satisfies $T(t,s) = T(t - s)$ for all $t \geq s \geq 0$, where for each $\phi = (\phi_1, \phi_2) \in L^p$, $t \to T(t)\phi$ is the solution $(u(t, \cdot), v(t, \cdot))$ to the linear parabolic system

$$\text{(LPS)} \qquad \begin{cases} u_t(t,x) = a\Delta u(t,x) & \\ v_t(t,x) = b\Delta v(t,x) & \text{for all } t > 0,\ x \in \Omega \\ u(0,x) = \phi_1(x), \quad v(0,x) = \phi_2(x) & \text{and } y \in \partial\Omega \\ u(t,y) = v(t,y) = 0 & \end{cases}$$

The nonlinear term B is the substitution operator generated by the map (f,g) of $\mathbb{R}^2$ into $\mathbb{R}^2$:

$$\begin{cases} [B\phi](x) = (f(\phi(x)), g(\phi(x))) \quad \text{for all } x \in \Omega \text{ and } \phi \in D(B) \\ \text{where } D(B) = \{\phi \in L^p : x \to (f(\phi(x)), g(\phi(x))) \text{ is in } L^p\} \end{cases}$$

The function $\alpha\colon [0,\infty) \to L^p$ reflects the inhomogeneous boundary conditions in (PS): $\alpha(t) = (\alpha_1(t, \cdot), \alpha_2(t, \cdot))$ where $\alpha_1,\alpha_2 : [0,\infty) \times \overline{\Omega} \to \mathbb{R}$ are continuous and satisfy

$$\begin{cases} \frac{\partial}{\partial t}\alpha_1(t,x) = a\Delta\alpha_1(t,x) & \\ \frac{\partial}{\partial t}\alpha_2(t,x) = b\Delta\alpha_2(t,x) & \text{for all } t > 0,\ x \in \Omega \\ & \text{and } y \in \partial\Omega \\ \alpha_1(t,y) = \beta(y), \quad \alpha_2(t,y) = \gamma(y) & \end{cases}$$

With T, B and α as above, consider the integral equation

$$\text{(IE-PS)} \quad (U(t), V(t)) = T(t)((u_o, v_o) - (\alpha_1(0, \cdot), \alpha_2(0, \cdot)) + (\alpha_1(t, \cdot), \alpha_2(t, \cdot)) + \int_o^t T(t - r)B(U(r), V(r))dr.$$

If (U,V) is a solution to (IE-PS) and $u(t,x) \equiv [U(t)](x)$ and $v(t,x) \equiv [V(t)](x)$ for all $t \geq 0$ and $x \in \Omega$, then (u,v) is called a mild L^p-solution to (PS). Theorem 2 may be applied to the integral equation (IE-PS), and hence give information on the behavior of solutions (or mild L^p-solutions) to (PS). We indicate the following two important examples:

Example 1. Suppose that $a = b$ and that Λ is a closed, bounded, convex subset of $\mathbb{R}^2$ such that $(u_o(x), v_o(x))$, $(\beta(y), \gamma(y)) \in \Lambda$ for all $x \in \Omega$ and $y \in \partial\Omega$. Suppose further that for each $(\xi_o, \eta_o) \in \Lambda$ there is a solution $t \to (\xi(t), \eta(t))$ to the ordinary differential equation

$$\text{(ODE)} \quad \begin{aligned} \xi'(t) &= f(\xi(t), \eta(t)) \qquad \xi(0) = \xi_o \\ \eta'(t) &= g(\xi(t), \eta(t)) \qquad \eta(0) = \eta_o \end{aligned}$$

such that $(\xi(t), \eta(t)) \in \Lambda$ for all $t \geq 0$ (i.e. Λ is positively invariant for (ODE)). Then (PS) has a mild L^p-solution (u,v) such that $(u(t,x), v(t,x)) \in \Lambda$ for all $t \geq 0$ and $x \in \Omega$. This result follows from Theorem 2 by taking $D = [0,\infty) \times K_p(\Lambda)$ where $K_p(\Lambda) = \{\phi \in L^p : \phi(x) \in \Lambda \text{ a.e. } x \in \Omega\}$.

In Example 1 it is crucial to require that $a = b$. However, using differential inequalities this requirement can be removed as is indicated by our second example.

Example 2. Suppose that $P = (P_1, P_2) : [0,\infty) \times \overline{\Omega} \to \mathbb{R}^2$ is an upper solution to (PS):

$$\begin{cases} \frac{\partial}{\partial t} P_1(t,x) \geq a\Delta P_1(t,x) + f(P(t,x)) & \\ \frac{\partial}{\partial t} P_2(t,x) \geq b\Delta P_2(t,x) + g(P(t,x)) & \text{for all } t > 0,\ x \in \Omega \\ & \text{and } y \in \partial\Omega \\ P_1(0,x) \geq u_o(x),\ \ P_2(0,x) \geq v_o(x) & \\ P_1(t,y) \geq \beta(y),\ \ P_2(t,y) \geq \gamma(y) & \end{cases}$$

and that $Q = (Q_1, Q_2) : [0,\infty) \times \overline{\Omega} \to \mathbb{R}^2$ is a lower solution to (PS) (that is, Q_1 and Q_2 satisfy analogous inequalities as P_1 and P_2 with "$\geq$" replaced by "$\leq$"). Suppose further that

$$\xi_1 = \eta_1 \text{ and } \xi_2 \geq \eta_2 \text{ implies } f(\xi_1, \xi_2) \geq f(\eta_1, \eta_2)$$

and

$$\xi_1 \geq \eta_1 \text{ and } \xi_2 = \eta_2 \text{ implies } g(\xi_1, \xi_2) \geq g(\eta_1, \eta_2).$$

Then there is a mild L^p-solution (u,v) to (PS) such that

$$Q_1(t,s) \leq u(t,x) \leq P_1(t,x) \text{ and } Q_2(t,x) \leq v(t,x) \leq P_2(t,x)$$

for all $t \geq 0$ and $x \in \Omega$.

This result follows from Theorem 2 by defining $D \subset [0,\infty) \times L^p$ by $(t,\phi) \in D$ only in case $t \geq 0$ and $Q_i(t,x) \leq \phi_i(x) \leq P_i(t,x)$ for $x \in \Omega$ and $i = 1,2$.

Further examples indicating the range of applicability of Theorem 2 and related abstract theorems may be found in [5, Chapter VIII and IX]. Moreover, in [2], an abstract theorem is presented that applies to (PS) when the gradients of the unknowns appear in the nonlinear terms. For example, consider the system

$$
(PS)' \quad \begin{cases} u_t(t,x) = a\Delta u(t,x) + f(u(t,x),\ v(t,x), \nabla u(t,x), \nabla v(t,x)) \\ v_t(t,x) = b\Delta v(t,x) + g(u(t,x),\ v(t,x), \nabla u(t,x), \nabla v(t,x)) \\ u(0,x) = u_o(x),\ v(0,x) = v_o(x) \\ u(t,y) = 0,\ v(t,y) = 0 \end{cases}
$$

where $t > 0$, $x \in \Omega$, and $y \in \partial\Omega$. Also, "∇" is the gradient with respect to the variable $x \in \Omega$, and hence f and g are continuous functions from $\mathbb{R}^2 \times \mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}$.

Example 3. In addition to the conditions in the above paragraph, suppose that $a = b$ and that there is a closed, bounded, convex subset Λ of $\mathbb{R}^2$ such that $(0,0) \in \Lambda$ and that there are numbers $L > 0$ and $\delta \in [0,2)$ such that

$$(*)\ |f(\xi_1, \xi_2, \eta_1, \eta_2)| \leq L(1 + |\eta_1| + |\eta_2|)^\delta \quad \text{for all} \quad \xi_1,\xi_2 \in \mathbb{R},\ \eta_1,\eta_2 \in \mathbb{R}^n.$$

Suppose further that if $\xi = (\xi_1, \xi_2) \in \Lambda$, $\eta = (\eta_i)_1^n$, $\zeta = (\zeta_i)_1^n \in \mathbb{R}^n$ and

$$\lim_{h\to 0} d(\xi + h(\sum_{i=1}^n \eta_i,\ \sum_{i=1}^n \zeta_i);\ \Lambda)/h = 0$$

then

$$\lim_{h\to 0+} d(\xi + h(f(\xi,\eta,\zeta),\ g(\xi,\eta,\zeta));\ \Lambda)/h = 0.$$

Then if $(u_o(x), v_o(x)) \in \Lambda$ for all $x \in \Omega$, there is a mild L^p-solution (u,v) to (PS)' on $[0,\infty) \times \Omega$ such that $(u(t,x), v(t,x)) \in \Lambda$ for all $t \geq 0$ and $x \in \Omega$.

The results of [2] can also be applied to obtain criteria for the stability of solutions to (PS)' as well (see Remark 7 of [2]). As an example, suppose that (*)

in Example 3 is satisfied and also that there are numbers $R_1, R_2 > 0$ and $\omega \geq 0$ such that if $(\xi_1, \xi_2) \in [0, R_1] \times [0, R_2]$ then

(a) if $|\xi_1| = \max\{|\xi_1|, |\xi_2|\}$ then $f(\xi_1, \xi_2, \theta, \zeta) \leq -\omega|\xi_1|$ for all $\zeta \in \mathbb{R}^n$,

(b) if $|\xi_2| = \max\{|\xi_1|, |\xi_2|\}$ then $g(\xi_1, \xi_2, \eta, \theta) \leq -\omega|\xi_2|$ for all $\eta \in \mathbb{R}^n$.

Then for each (u_o, v_o) such that $(u_o(x), v_o(x)) \in [0, R_1] \times [0, R_2]$ there is a mild L^p-solution (u,v) to (PS)' on $[0,\infty) \times \Omega$ such that

$$\operatorname*{ess\,sup}_{x \in \Omega} \max\{|u(t,x)|, |v(t,x)|\} \leq e^{-\omega t} \operatorname*{ess\,sup}_{x \in \Omega} \max\{|u_o(x)|, |v(x)|\}$$

for all $t \geq 0$.

As one final comment we remark that the techniques used in each of our examples apply to systems of m equations and m unknowns for any integer $m \geq 1$.

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